

SCALAR TREFOILS

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Abstract. The knot model is extended by assuming that the trefoils are realized as either chiral fermions or as scalar bosons. There are then four scalar trefoils with electric charges $(0, -1, 2/3, -1/3)$ that may be classified in the same way as the chiral fermions: as two isotopic doublets where the two doublets have different hypercharge and the two members of the doublets have different t_3 . Only the neutral scalar plays the role of the standard Higgs in fixing the mass ratios of the vector bosons, while the charged scalars, in addition to having the usual electromagnetic interactions of scalar particles, fix the mass spectrum of the fermions. The extended model would suggest a search for the charged scalars.

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1 Introduction.

In view of the existence and topological stability of knots in classical field theory, it is natural to explore the role of the same structures in quantum field theory and in particular to examine the possibility of interpreting the field quanta of the standard theory as knotted solitons instead of point particles. When the standard point particles are replaced by the knots, one finds that the topological structure of the fermionic knot is determined by its electric charge, hypercharge, and isotopic spin.^{1,2,3} This result permits a geometric interpretation of charge, i-spin, and hypercharge and thereby also permits one to assign a knot structure to the vector bosons. Among other consequences it then also becomes natural to expand the Higgs sector so that there is a multiplet of 4 scalars, instead of a single Higgs, with the same charges as the fermions.

2 The Knot Conjecture.

If the elementary particles are quantum knots, one expects that the simplest knots (trefoils) are realized as the most elementary particles (chiral fermions and scalar bosons). To test this idea we may start from the observation that there are 4 trefoils and 4 families of elementary fermions (neutrinos, leptons, up quarks, down quarks). We shall assume that the 3 members of each family (e, μ, τ) , $(\nu_e, \nu_\mu, \nu_\tau)$, (u, c, t) , (d, s, b) are the three lowest states of the 4 fermionic solitons.

We shall now also assume that there are 4 scalar knots as well as 4 spinor knots. Then the complete wave function of the spinor or the scalar soliton is the product of the standard chiral spinor or scalar wave function multiplied by a knot factor. The knot factor depends on the symmetry algebra of the knot, which is $SU_q(2)$, and we take the quantum state of the knot to be determined by an irreducible representation of $SU_q(2)$, denoted by $D_{mm'}^j$. We shall now assume that the knot factor is the quantum state of the knot,³ namely

$$D_{mm'}^j = D_{\frac{w}{2} \frac{r+1}{2}}^{N/2} \quad \text{or equivalently} \quad D_{-3t_3-3t_0}^{3t} \quad (2.1)$$

with

$$\begin{aligned}
N &= 6t \\
w &= -6t_3 \\
r+1 &= -6t_0
\end{aligned} \tag{2.2}$$

where $D_{mm'}^j$ is a matrix element of the $2j+1$ irreducible representation of the quantum group $SU_q(2)$. Here N, w , and r are the number of crossings, the writhe, and the rotation of the knot while t, t_3 , and t_0 are i-spin, 3-component of the i-spin, and hypercharge. For the spinor and scalar realization of the trefoil one has $N = 3$ while the 4 separate realizations of the trefoil are described in Table 1.

	t	t_3	t_0	w	r	Q	$D_{mm'}^{3/2}$
ν	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-3	2	0	$D_{-\frac{3}{2}+\frac{3}{2}}^{3/2}$
e^-	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	3	2	$-e$	$D_{\frac{3}{2}\frac{3}{2}}^{3/2}$
u	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$	-3	-2	$\frac{2}{3}e$	$D_{-\frac{3}{2}-\frac{1}{2}}^{3/2}$
d	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{6}$	3	-2	$-\frac{1}{3}e$	$D_{\frac{3}{2}-\frac{1}{2}}^{3/2}$

Table 1.

In the table the first three columns describe the standard description of the four classes of fermions (ν, e, u, d) and the next three columns describe the writhe, rotation and electric charge of the same classes regarded as knots. In the last column is the corresponding irreducible representation where $m = \frac{w}{2}$ and $m' = \frac{r+1}{2}$. For a detailed justification of this table and Eq. (2.2), see Ref. 3. We are now assuming that there is a single table for scalars and chiral fermions.

By (2.1) and (2.2) one has Table 2 for the 4 vector bosons.

	t	t_3	t_0	$D_{mm'}^j$
W^+	1	1	0	D_{-30}^3
W^-	1	-1	0	D_{30}^3
W^3	1	0	0	D_{00}^3
W^0	0	0	0	D_{00}^0

Table 2.

We shall abbreviate the knot factor (the quantum state of the knot) as follows:

$$\text{For spinor and scalar fields write } D_k \quad k = (\nu, e, u, d) \quad (2.3)$$

$$\text{For vector bosons write } \mathcal{D}_\alpha \quad \alpha = (+, -, 3, 0) \quad (2.4)$$

Since the $D_{mm'}^j$ are irreducible representations of $SU_q(2)$ we shall next collect the relevant information about $SU_q(2)$.

3 The Knot Algebra.

One way of seeing that $SL_q(2)$ is the appropriate algebra of the knot is to observe on the one hand that the Kauffman algorithm (for generating the Kauffman or the Jones polynomial that characterizes a knot) may be expressed in terms of the matrix

$$\epsilon_q = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix} \quad \epsilon_q^2 = -1 \quad (3.1)$$

and on the other hand that ϵ_q is also the invariant matrix of $SL_q(2)$ since

$$T^t \epsilon_q T = T \epsilon_q T^t = \epsilon_q \quad (3.2)$$

where T belongs to a two-dimensional representation of $SL_q(2)$.

We shall now describe this algebra. Let

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (3.3)$$

Then by (3.2) the matrix elements of T satisfy the following algebra

$$\begin{aligned}
ab &= qba & bd &= qdb & ad - qbc &= 1 & bc &= cb \\
ac &= qca & cd &= qdc & da - q_1cb &= 1 & q_1 &= q^{-1}
\end{aligned} \tag{A}$$

In the discussion of electroweak we need only the unitary subalgebra obtained by setting

$$\begin{aligned}
d &= \bar{a} \\
c &= -q_1 \bar{b}
\end{aligned}$$

Then (A) reduces to the following

$$\begin{aligned}
ab &= qba & a\bar{a} + b\bar{b} &= 1 & b\bar{b} &= \bar{b}b \\
a\bar{b} &= q\bar{b}a & \bar{a}a + q_1^2 \bar{b}b &= 1
\end{aligned} \tag{A}'$$

For the physical applications we need the higher representations of $SU_q(2)$. The $2j + 1$ -dimensional unitary irreducible representations of the $SU_q(2)$ algebra (A)' are

$$D_{mm'}^j = \sum_{s,t} A_{mm'}^j(s,t) \delta(s+t, n'_+) a^s b^{n_+-s} \bar{b}^t \bar{a}^{n_--t} \tag{3.4}$$

where

$$A_{mm'}^j(s,t) = \left[\frac{\langle n'_+ \rangle_1! \langle n'_- \rangle_1!}{\langle n_+ \rangle_1! \langle n_- \rangle_1!} \right]^{1/2} \left\langle \begin{matrix} n_+ \\ s \end{matrix} \right\rangle_1 \left\langle \begin{matrix} n_- \\ t \end{matrix} \right\rangle_1 q^{t(n_+-s+1)} (-)^t$$

and

$$\begin{aligned}
n_{\pm} &= j \pm m & \left\langle \begin{matrix} n \\ s \end{matrix} \right\rangle_1 &= \frac{\langle n \rangle_1!}{\langle s \rangle_1! \langle n-s \rangle_1!} & \langle n \rangle_1 &= \frac{q_1^{2n} - 1}{q_1^2 - 1} \\
n'_{\pm} &= j \pm m'
\end{aligned}$$

The algebra (A)' is invariant under the gauge transformations

$$\begin{aligned}
a' &= e^{i\varphi_a} a & b' &= e^{i\varphi_b} b \\
\bar{a}' &= e^{-i\varphi_a} \bar{a} & \bar{b}' &= e^{-i\varphi_b} \bar{b}
\end{aligned} \tag{3.5}$$

These transformations induce the following gauge transformations on $D_{mm'}^j$.³

$$D_{mm'}^{j'} = U_a U_b D_{mm'}^j \tag{3.6}$$

where

$$\begin{aligned}
U_a &= e^{i\varphi_a(m+m')} \\
U_b &= e^{i\varphi_b(m-m')}
\end{aligned} \tag{3.7}$$

and by (2.1) and (2.2)

$$\begin{aligned} m + m' &= -\frac{1}{2} (w + r + 1) = -3(t_3 + t_0) \\ m - m' &= -\frac{1}{2} (w - r - 1) = -3(t_3 - t_0) \end{aligned} \quad (3.8)$$

Since b and \bar{b} commute, they have common eigenstates. Let $|0\rangle$ be designated as a ground state and let

$$b|0\rangle = \beta|0\rangle \quad (3.9)$$

$$\bar{b}|0\rangle = \beta^*|0\rangle \quad (3.10)$$

and

$$\bar{b}b|0\rangle = |\beta|^2|0\rangle \quad (3.11)$$

where $\bar{b}b$ is Hermitian with real eigenvalues and orthogonal eigenstates.

One finds by $(A)'$ that

$$\bar{b}b|n\rangle = E_n|n\rangle \quad (3.12)$$

where

$$|n\rangle \sim \bar{a}^n|0\rangle \quad (3.13)$$

and

$$E_n = q^{2n}|\beta|^2 \quad (3.14)$$

$\bar{b}b$ resembles the Hamiltonian of an oscillator but with eigenvalues arranged in geometrical progression and with $|\beta|^2$ corresponding to $\frac{1}{2} \hbar\omega$. If we take $H(\bar{b}b)$ to be the Hamiltonian of the knot, where the functional form of H is left unspecified, it will have the same eigenstates.

4 The Normal Modes of the Quantum Fields.

We assume that the separate fermion and scalar states are the low lying states of the fermion and scalar solitons as follows:

$$D_k|n\rangle \quad k = (\nu, \ell, u, d), \quad n = (0, 1, 2) \quad (4.1)$$

by (2.3), and where $|n\rangle$ is the n^{th} level of the “ q -oscillator” given by (3.13). Then the complete L -chiral normal modes of the Dirac fields are

$$L_{kn} = \chi_{kn}(\vec{p}, \vec{t}) D_k |n\rangle \quad (4.2)$$

The corresponding normal modes of the scalar fields are

$$\Phi_{kn} = \varphi_{kn}(\vec{p}, \vec{t}) D_k |n\rangle \quad (4.3)$$

Here χ_{kn} and φ_{kn} are normal modes of the standard Dirac and of the hypothetical scalar fields, respectively.

The standard theory is invariant under local $SU(2)$ and the fermions and scalars are doublets under $SU(2)$. The two members of the doublet differ in t_3 (or in the writhe of the knot), while the (ν, ℓ) doublets differ from the (u, d) doublets in the hypercharge (or in the rotation of the knot). Therefore we have the two doublet solitons differing in hypercharge (or rotation).

$$\begin{pmatrix} \chi_\nu D_\nu \\ \chi_\ell D_\ell \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \chi_u D_u \\ \chi_d D_d \end{pmatrix} \quad (4.4)$$

The vector connection in the standard theory is

$$W_+ t_+ + W_- t_- + W_3 t_3 + W_0 t_0 \quad (4.5)$$

where (t_+, t_-, t_3, t_0) are the generators of the standard electroweak theory in the charge representation and

$$t_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad t_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad t_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (4.6)$$

We now replace (4.5) by

$$W_+ \tau_+ + W_- \tau_- + W_3 \tau_3 + W_0 \tau_0 \quad (4.7)$$

where

$$\tau_k = c_k(q, \beta) t_k \mathcal{D}_k \quad k = (+, -, 3, 0) \quad (4.8)$$

and the \mathcal{D}_k are the charge states of the four vector mesons

$$\mathcal{D}_+ \equiv D_{-30}^3/N_+ = \bar{b}^3\bar{a}^3 \quad (4.9)$$

$$\mathcal{D}_- \equiv D_{03}^3/N_- = a^3b^3 \quad (4.10)$$

$$\mathcal{D}_3 \equiv D_{00}^3 = f_3(\bar{b}b) \quad (4.11)$$

$$\mathcal{D}_0 \equiv D_{00}^0 = 1 \quad (4.12)$$

where $f_3(\bar{b}b)$ is computed from (3.4) and where N_+ and N_- are numerical factors following from (3.4).

The c_k are numerical functions of the parameters (q, β) and are partially fixed by relations between the masses of the vector bosons. The c_k are also proportional to the coupling constants and these are different for \vec{W} and W_0 .

For the vector connection we shall set

$$\mathcal{W}_\mu = W_\mu^\alpha \tau_\alpha \quad (4.13)$$

Then the covariant derivative is

$$\nabla = 1 \partial + \mathcal{W} \quad (4.14)$$

5 Quantum Field Theory.

The quantum field theory is defined by a Lagrangian invariant under a group of local gauge transformations and by boundary conditions defined by the ground states of the fields. The fields and states are defined only up to a gauge transformation. To select the ground state from the ensemble of gauge equivalent ground states we select a special gauge. This privileged gauge, along with both the Fock vacuum and the lowest state of the q -algebra, limits the ground state of the model which will now be denoted by $|0\rangle\rangle$,

We shall further define the ground state by the requirement that all the interacting fields be independent of position in this state. Then all kinetic energy terms in this state will vanish. The constant fields shall also be chosen to minimize the potential so that the total energy will be minimized for this vacuum state.

Now define the Lorentz invariant scalar products $\bar{\varphi}_k \varphi_k$ and $\bar{W}_\alpha^\mu W_{\alpha\mu}$ by

$$\bar{\varphi}_k \varphi_k = \langle\langle 0 | \bar{\Phi}_k \Phi_k | 0 \rangle\rangle \quad k = \nu, \ell, u, d \quad (5.1)$$

$$\bar{W}_\alpha^\mu W_{\alpha\mu} = \langle\langle 0 | \bar{W}_\alpha^\mu W_{\alpha\mu} | 0 \rangle\rangle \quad \alpha = (+, -, 3, 0) \quad (5.2)$$

where Φ_k and W_α^μ are the position independent fields defined over the ground state and both sides of these equations are to be taken in the privileged gauge. In the Goldstone-Higgs language the Goldstone fields, which are parameters of the gauge transformation, are gauged to vanish in the privileged gauge. In the same gauge the ground states of four scalar solitons are arranged in two doublets that take the following form:

$$\begin{pmatrix} \phi_\nu D_\nu | 0 \rangle \\ \phi_\ell D_\ell | 0 \rangle \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \phi_u D_u | 0 \rangle \\ \phi_d D_d | 0 \rangle \end{pmatrix} \quad (5.3)$$

labelled by hypercharge $t_0 = -1/2$ and $t_0 = 1/6$ (or by rotation $r = 2$ and $r = -2$) respectively. The three Goldstone degrees of freedom associated with $SU(2)$ gauge transformations are expressed not as Goldstone bosons, but as the three degrees of freedom associated with the longitudinal modes of the three massive vectors in the standard way.

6 Masses of the Vectors.

The invariant form of the kinetic energy of each scalar doublet is

$$\frac{1}{2} \text{Tr} \bar{\nabla}_\mu \Phi_k \nabla^\mu \Phi_k \quad (6.1)$$

where k signifies the hypercharge (or the rotation of the knot). Here

$$\nabla_\mu = \partial_\mu + W_\mu^\alpha \tau_\alpha \quad (6.2)$$

and Φ_k is one of the doublets described by (5.3).

Let us write

$$\Phi_k = \begin{pmatrix} u_k \\ v_k \end{pmatrix} \quad (6.3)$$

and let us consider $k = t_0 = -1/2$. Then

$$\Phi(t_0 = -1/2) = \begin{pmatrix} \varphi_\nu D_\nu | 0 \rangle \\ \varphi_\ell D_\ell | 0 \rangle \end{pmatrix} \quad (6.4)$$

and

$$\begin{aligned} \nabla_\mu \Phi \left(-\frac{1}{2} \right) = \partial_\mu \begin{pmatrix} \varphi_\nu D_\nu |0\rangle \\ \varphi_\ell D_\ell |0\rangle \end{pmatrix} + \begin{pmatrix} W_\mu^+ \mathcal{D}_+ \varphi_\ell D_\ell |0\rangle \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ W_\mu^- \mathcal{D}_- \varphi_\nu D_\nu |0\rangle \end{pmatrix} \\ + \text{neutral couplings} \begin{pmatrix} \varphi_\nu D_\nu |0\rangle \\ \varphi_\ell D_\ell |0\rangle \end{pmatrix} \end{aligned} \quad (6.5)$$

The neutral couplings, written in terms of the physical fields, A and Z , are

$$i g W_\mu^3 \tau_3 + i g_0 W_\mu^0 \tau_0 = \mathcal{A} A_\mu + \mathcal{Z} Z_\mu \quad (6.6)$$

where

$$\mathcal{A} = i(g\tau_3 \sin \theta + g_0\tau_0 \cos \theta) = ie(\tau_3 + \tau_0) \quad (6.7)$$

$$\mathcal{Z} = i(g\tau_3 \cos \theta - g_0\tau_0 \sin \theta) = ie(\cot \theta \tau_3 - \tan \theta \tau_0) \quad (6.8)$$

and θ is the Weinberg angle defined by

$$\sin \theta = \frac{e}{g} \quad \text{or} \quad \cos \theta = \frac{e}{g_0} \quad (6.9)$$

Note that

$$\mathcal{A}|\nu\rangle = 0 \quad (6.10)$$

if $|\nu\rangle$ is any neutral state and therefore by (6.7)

$$(\tau_0 + \tau_3)|\nu\rangle = 0 \quad (6.11)$$

Then by (6.8) and the preceding equation

$$\mathcal{Z}|\nu\rangle = ie(\cot \theta + \tan \theta)\tau_3|\nu\rangle$$

or

$$\mathcal{Z}|\nu\rangle = \frac{ig}{\cos \theta} \tau_3|\nu\rangle \quad (6.12)$$

Then by (6.6)-(6.8)

$$[\text{neutral couplings}] \begin{pmatrix} \varphi_\nu D_\nu |0\rangle \\ \varphi_\ell D_\ell |0\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{A}_\ell \varphi_\ell D_\ell |0\rangle \end{pmatrix} A_\mu + \begin{pmatrix} \mathcal{Z}_\nu \varphi_\nu D_\nu |0\rangle \\ \mathcal{Z}_\ell \varphi_\ell D_\ell |0\rangle \end{pmatrix} Z_\mu \quad (6.13)$$

where by (4.8)

$$\mathcal{A}_\ell|\ell\rangle = ie(c_3 t_3 \mathcal{D}_3 + c_0 t_0 \mathcal{D}_0)|\ell\rangle$$

The kinetic energy (6.1) may now be written as

$$\begin{aligned} & \langle 0|\bar{D}_\nu\partial_\mu\bar{\varphi}_\nu\cdot\partial^\mu\varphi_\nu D_\nu|0\rangle + \langle 0|\bar{D}_\ell\partial_\mu\bar{\varphi}_\ell\cdot\partial^\mu\varphi_\ell D_\ell|0\rangle \\ & + \langle 0|\bar{D}_\ell\bar{\mathcal{D}}_+\mathcal{D}_+D_\ell|0\rangle\bar{\varphi}_\ell\varphi_\ell\bar{W}^{+\mu}W_\mu^+ + \langle 0|\bar{D}_\nu\bar{\mathcal{D}}_-\mathcal{D}_-D_\nu|0\rangle\bar{\varphi}_\nu\varphi_\nu\bar{W}^{-\mu}W_\mu^- \\ & + \langle 0|\bar{D}_\ell\bar{\mathcal{A}}_\ell\mathcal{A}_\ell D_\ell|0\rangle\bar{\varphi}_\ell\varphi_\ell A^\mu A_\mu + \langle 0|\bar{D}_\nu\bar{\mathcal{Z}}_\nu\mathcal{Z}_\nu D_\nu|0\rangle\bar{\varphi}_\nu\varphi_\nu Z^\mu Z_\mu \\ & + 0 + \langle 0|\bar{D}_\ell\bar{\mathcal{Z}}_\ell\mathcal{Z}_\ell D_\ell|0\rangle\bar{\varphi}_\ell\varphi_\ell Z^\mu Z_\mu \end{aligned} \quad (6.14)$$

The part of the kinetic energy contributing to the total Lagrangian and coming from the ν -scalar is

$$I_\nu\partial_\mu\bar{\varphi}_\nu\partial^\mu\varphi_\nu + I_{\nu--}\bar{\varphi}_\nu\varphi_\nu(W_1^\mu W_{1\mu} + W_2^\mu W_{2\mu}) + I_{\nu\mathcal{Z}\mathcal{Z}}\bar{\varphi}_\nu\varphi_\nu Z^\mu Z_\mu \quad (6.15)$$

where

$$I_\nu = \frac{1}{2} \text{Tr} \langle 0|\bar{D}_\nu D_\nu|0\rangle \quad (6.16)$$

$$I_{\nu--} = \frac{1}{2} \text{Tr} \langle 0|\bar{D}_\nu\bar{\mathcal{D}}_-\mathcal{D}_-D_\nu|0\rangle \quad (6.17)$$

$$I_{\nu\mathcal{Z}\mathcal{Z}} = \frac{1}{2} \text{Tr} \langle 0|\bar{D}_\nu\bar{\mathcal{Z}}_\nu\mathcal{Z}_\nu D_\nu|0\rangle \quad (6.18)$$

The part of the kinetic energy contributing to the total Lagrangian and coming from the ℓ -scalar is

$$\begin{aligned} & I_\ell\partial_\mu\bar{\varphi}_\ell\partial^\mu\varphi_\ell + I_{\ell++}(W_1^\mu W_{1\mu} + W_2^\mu W_{2\mu})\bar{\varphi}_\ell\varphi_\ell \\ & + I_{\ell aa}\bar{\varphi}_\ell\varphi_\ell A^\mu A_\mu + I_{\ell\mathcal{Z}\mathcal{Z}}\bar{\varphi}_\ell\varphi_\ell Z^\mu Z_\mu \end{aligned} \quad (6.19)$$

where

$$I_\ell = \frac{1}{2} \text{Tr} \langle 0|\bar{D}_\ell D_\ell|0\rangle \quad (6.20)$$

$$I_{\ell++} = \frac{1}{2} \text{Tr} \langle 0|\bar{D}_\ell\bar{\mathcal{D}}_+\mathcal{D}_+D_\ell|0\rangle \quad (6.21)$$

$$I_{\ell aa} = \frac{1}{2} \text{Tr} \langle 0|\bar{D}_\ell\bar{\mathcal{A}}_\ell\mathcal{A}_\ell D_\ell|0\rangle \quad (6.22)$$

$$I_{\ell\mathcal{Z}\mathcal{Z}} = \frac{1}{2} \text{Tr} \langle 0|\bar{D}_\ell\bar{\mathcal{Z}}_\ell\mathcal{Z}_\ell D_\ell|0\rangle \quad (6.23)$$

In interpreting these terms set

$$\varphi_\ell = \overset{\circ}{\varphi}_\ell + u_\ell \quad (6.24)$$

$$\varphi_\nu = \overset{\circ}{\varphi}_\nu + u_\nu \quad (6.25)$$

where $\overset{\circ}{\varphi}_\ell$ and $\overset{\circ}{\varphi}_\nu$ are the vacuum expectation values of φ_ℓ and φ_ν defined by (5.1).

Now we make the basic assumptions

$$\overset{\circ}{\varphi}_\ell = 0 \quad (6.26)$$

$$\overset{\circ}{\varphi}_\nu = \rho_\nu \neq 0 \quad \bar{\rho}_\nu = \rho_\nu \quad (6.27)$$

The effective Lagrangian of the ν -scalar coming from (6.15) is

$$I_\nu \partial_\mu \bar{u}_\nu \partial^\mu u_\nu \quad (6.28)$$

while W^+ , W^- , and Z obtain masses from the following terms of (6.15).

$$I_{\nu--} \rho_\nu^2 (W_1^\mu W_{1\mu} + W_2^\mu W_{2\mu}) + I_{\nu ZZ} \rho_\nu^2 Z_\mu Z^\mu \quad (6.29)$$

The effective Lagrangian of the ℓ -scalar is on the other hand by (6.19)

$$\begin{aligned} I_\ell \partial_\mu \bar{u}_\ell \partial^\mu u_\ell &+ I_{\ell++} (W_1^\mu W_{1\mu} + W_2^\mu W_{2\mu}) \bar{u}_\ell u_\ell \\ &+ I_{\ell aa} \bar{u}_\ell u_\ell A_\mu A^\mu + I_{\ell ZZ} \bar{u}_\ell u_\ell Z_\mu Z^\mu \end{aligned} \quad (6.30)$$

The terms in (6.29) that contribute to the W^+ , W^- and Z mass now appear in (6.30) as interaction terms between the same vector fields and the ℓ -scalar.

In discussing the $t_0 = \frac{1}{6}$ doublet we assume as in (6.26)

$$\overset{\circ}{\varphi}_u = \overset{\circ}{\varphi}_d = 0 \quad (6.31)$$

and therefore obtain a result analogous to (6.30) rather than (6.28) and (6.29).

If we had assumed $\overset{\circ}{\varphi}_\ell \neq 0$, then the vacuum state would have been charged and the photon field would have had a mass by (6.30) with u_ℓ replaced by $\overset{\circ}{\varphi}_\ell$. The same remark requires Eq. (6.31).

According to the preceding assumptions only the neutral member of the trefoil scalar quadruplet supplies mass to the vector bosons. These masses depend on the coefficients I_ν , $I_{\nu--}$ and $I_{\nu ZZ}$ that are given by (6.16)-(6.18), and are functions of q and β . The numerical values of q and β , and therefore the c_k , defined in (4.8), are limited by the empirical ratios of the masses of the vector bosons (as discussed in Ref. 2).

7 Masses of the Fermions.

According to the standard theory the fermion mass term is

$$\mathcal{M} = \frac{m}{\rho}(\bar{\psi}^L \varphi \psi^R + \bar{\psi}^R \bar{\varphi} \psi^L) \quad (7.1)$$

where ρ is the vacuum expectation value of the Higgs scalar and both ψ^R and the product $\bar{\psi}^L \varphi$ are SU(2) singlets. The φ doublet is composed of a neutral and a charged component. To examine the mass in the standard theory one chooses the unitary gauge in which the charged component is rotated away. In order to represent the masses of the individual fermions a numerical mass matrix is then introduced.

In the present model we consider the following modification of (7.1).

$$\mathcal{M}_n = \sum_{t_0} (\bar{\psi}_n^L(t_0) \Phi_n(t_0) \psi_n^R(t_0) + \bar{\psi}_n^R(t_0) \bar{\Phi}_n(t_0) \psi_n^L(t_0)) \quad (7.2)$$

where $\Phi_n(t_0)$ is an SU(2) doublet and $\bar{\psi}^L \Phi(t_0)$ is again a SU(2) singlet to maintain SU(2) gauge invariance. The sum is over the two values, $(-1/2)$ and $(1/6)$, of t_0 .

Here t_0 is the hypercharge and we define

$$\Phi_n \left(-\frac{1}{2} \right) = \begin{pmatrix} \rho_\nu D_\nu |n\rangle \\ \rho_\ell D_\ell |n\rangle \end{pmatrix} \quad (7.3)$$

$$\Phi_n \left(\frac{1}{6} \right) = \begin{pmatrix} \rho_u D_u |n\rangle \\ \rho_d D_d |n\rangle \end{pmatrix} \quad (7.4)$$

and

$$\psi_n^L \left(-\frac{1}{2} \right) = \begin{pmatrix} \chi_\nu D_\nu |n\rangle \\ \chi_\ell D_\ell |n\rangle \end{pmatrix} \quad (7.5)$$

$$\psi_n^L \left(\frac{1}{6} \right) = \begin{pmatrix} \chi_u D_u |n\rangle \\ \chi_d D_d |n\rangle \end{pmatrix} \quad (7.6)$$

and

$$\bar{\rho}_k = \rho_k \quad k = \nu, \ell, u, d \quad (7.7)$$

Here ρ_k is the vacuum expectation value of φ_k if $k = \nu$ but not if $k = (\ell, u, d)$. In the latter cases the ρ_k like the c_k in (4.8) are to be understood as numerical scaling parameters that may be required to fix the relative masses of the four trefoils.

Note that the neutral scalar now plays a role in fixing the masses of the neutral fermions as well as the masses of the neutral bosons. On the other hand the charged scalars play quite different roles in the two cases. In the bosonic case the charged scalars are spacetime fields that play no role in determining the bosonic masses. In the fermion mass formula (7.2), however, they do not enter as spacetime fields but as their “internal wave functions”, the trefoil factors, $D_{\frac{m}{2} \frac{r+1}{2}}^{3/2}$, lying in the knot algebra.

With these assumptions let us expand the $t_0 = -\frac{1}{2}$ part of (7.2) as follows:

$$\mathcal{M}_n \left(-\frac{1}{2} \right) = \langle n | \bar{D}_\nu \chi_\nu^L(n) \rho_\nu D_\nu + \bar{D}_\ell \bar{\chi}_\ell^L(n) \rho_\ell D_\ell | n \rangle \chi^R + \text{adjoint} \quad (7.8)$$

$$\begin{aligned} &= \rho_\nu \langle n | \bar{D}_\nu D_\nu | n \rangle (\bar{\chi}_\nu^L(n) \chi^R + \bar{\chi}^R \chi_\nu^L(n)) \\ &\quad + \rho_\ell \langle n | \bar{D}_\ell D_\ell | n \rangle (\bar{\chi}_\ell^L(n) \chi^R + \bar{\chi}^R \chi_\ell^L(n)) \end{aligned} \quad (7.9)$$

$\chi_\ell(n)$ is the left chiral component of $\psi_\ell(n)$

$$\chi^L(n) = \frac{1}{2} (1 - \gamma_5) \psi_\ell(n) \quad (7.10)$$

χ^R transforms like the right chiral component of a Dirac spinor, which we may now take to be $\psi_\ell(n)$:

$$\chi^R(n) = \frac{1}{2} (1 + \gamma_5) \psi_\ell(n) \quad (7.11)$$

Then by (7.9)

$$\mathcal{M}_n \left(-\frac{1}{2} \right) = m_\nu(n) \bar{\chi}_\nu(n) \chi_\nu(n) + m_\ell(n) \bar{\chi}_\ell(n) \chi_\ell(n) \quad (7.12)$$

where

$$m_\nu(n) = \rho_\nu \langle n | \bar{D}_\nu D_\nu | n \rangle \quad (7.13)$$

$$m_\ell(n) = \rho_\ell \langle n | \bar{D}_\ell D_\ell | n \rangle \quad (7.14)$$

One obtains similar results for the quark doublet:

$$m_u(n) = \rho_u \langle n | \bar{D}_u D_u | n \rangle \quad (7.15)$$

$$m_d(n) = \rho_d \langle n | \bar{D}_d D_d | n \rangle \quad (7.16)$$

These results may be summarized in the following expressions for the masses of the elementary fermions

$$M_n(w, r) = \rho(w, r) \langle n | \bar{D}_{\frac{w}{2} \frac{r+1}{2}}^{3/2} D_{\frac{w}{2} \frac{r+1}{2}}^{3/2} | n \rangle \quad (7.17)$$

or

$$M_n(t_3, t_0) = \rho(t_3, t_0) \langle n | \bar{D}_{-3t_3}^{3/2} D_{-3t_0}^{3/2} | n \rangle \quad (7.18)$$

These relations have been discussed in Refs. 1 and 2. Again the values of q and β need to be adjusted to agree with the empirical mass ratios, as illustrated in Refs. 1 and 2.

Since there are two parameters, q and β , and since there appear to be three masses in each family, ratios of the masses may be computed without ambiguity. To determine absolute masses one needs the prefactor $\rho(t_3, t_0)$.

If there are in fact only three masses in each family, the present model needs to be supplemented by an exclusion principle allowing only three masses.

8 Remarks.

The starting point of the considerations leading to a knot model of the elementary particles is the fact that knotted field configurations appear in certain classical field theories and that they are also topologically stable. These facts suggest the investigation of a knot model in which the knot is characterized solely by the algebra of the knot, $SU_q(2)$, independent of its particular field realization. If this model of the elementary particles has physical content, one would expect the simplest particles to be the simplest knots (trefoils). Consistent with this idea is the fact that there are four trefoils and four classes of elementary fermions. Moreover there is a unique correspondence between the four trefoils and the four classes that is made possible by the structure of the knot on the one hand and on the other hand by the spectrum of charges $(0, -e, 2/3e, -e/3)$ displayed by the four classes of fermions.³ This correspondence is established in Ref. 3 and stated in (2.2).

It is a simple formal extension of the conjecture that the trefoils are realized as the lowest fermionic representations of the Lorentz group, to assume that they are also realized as the lowest bosonic representations of the Lorentz group. Then there would be four Higgs-like scalars and suggests that the search for scalars be widened.

If these particles are associated with minima of a Higgs-like potential, the potential would have to be defined over the $SU_q(2)$ algebra. One may define such a potential, for

example, by

$$\langle 0|V(\bar{\varphi}\varphi)|0\rangle$$

where $|0\rangle$ is the ground state of the q -algebra. This expectation value may be so defined that the minima lie at the trefoil points of the algebra, i.e., at the four points represented by $\varphi = D^{\frac{N/2}{\frac{w}{2}\frac{r+1}{2}}}$ where (w, r) describes a trefoil.

The present model depends on two parameters, q and β . In comparing this model with empirical data, such as the mass ratios of the vector bosons or the mass ratios of the elementary fermions, the values of q and β have been found to fluctuate.² That is to be expected since the knot model as here presented is based on only the standard electroweak theory and ignores gluon and gravitational interactions. The parameters, q and β , may be regarded as surrogates for the neglected interactions or as simply the parameters of an effective Lagrangian.

Since the model is characterized by the realization of the trefoil as either of the two lowest representations of the Lorentz group, namely the chiral fermion and the scalar boson, a supersymmetric extension is naturally suggested.

References.

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3. R. J. Finkelstein, "The Elementary Particles as Quantum Knots in Electroweak Theory", hep-th/0705.3656.